Averages along the squares on the torus

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Abstract

Answering a question raised by J-P. Conze we show that for any $x, \alpha \in \mathbb{T}$, $\alpha \not\in \mathbb{Q}$ there exist $f \in L^1(\mathbb{T})$, $f \geq 0$ such that the averages

\[ (\star) \quad \frac{1}{N} \sum_{n=1}^{N} f(y + nx + n^2 \alpha) \]

diverge for a.e. $y$. By Birkhoff’s Ergodic Theorem applied on $\mathbb{T}^2$ for the transformation $(x, y) \mapsto (x + \alpha, y + 2x + \alpha)$ for almost every $x \in \mathbb{T}$ the averages $(\star)$ converge for a.e. $y$. We show that given $\alpha \not\in \mathbb{Q}$ one can find $f \in L^1(\mathbb{T})$ for which the set $D_{\alpha, f} = \{ x \in \mathbb{T} : (\star) \text{ diverges for a.e. } y \text{ as } N \to \infty \}$ is of Hausdorff dimension one. We also show that for a polynomial $p(n)$ of degree two with integer coefficients the sequence $p(n)$ is universally $L^1$-bad.